

Boundary problems for one-dimensional kinetic equation with constant collision frequency

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Abstract

For the one-dimensional linear kinetic equation analytical solutions of problems about temperature jump and weak evaporation (condensation) over flat surface are received. The equation has integral of collisions BGK (Bhatnagar, Gross and Krook) and constant frequency of collisions of molecules. Distribution of concentration, mass speed and temperature is received.

Key words: kinetic equation, frequency of collisions, preservation laws, separation of variables, characteristic equation, dispersion equation, eigenfunctions, analytical solution, boundary problems.

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Introduction

In work [1] the linear one-dimensional kinetic equation with integral of collisions BGK (Bhatnagar, Gross and Krook) and frequency of collisions, affine depending on the module velocity of molecules has been entered. Preservation laws of numerical density (concentration) of molecules, momentum of molecules and energy have been thus used.

In [1] the theorem about structure of general solution of the entered equation has been proved.

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In work [2], being continuation of [1], are received exact solutions of the problem on temperature jump and weak evaporation (condensation) in rarefied gas for kinetic equation with frequency of collisions, affine depending on the module velocity of molecules.

In the present work which is continuation of [1] and [2], exact solutions of the problem about temperature jump and weak evaporation (condensation) in the rarefied gas are received. Here the one-dimensional equation with constant frequency of collisions is used. This equation is a special case of the kinetic equation with frequency of collisions, affine depending on the module velocity of molecules.

These two problems following [3] we will name the generalized Smoluchowsky' problem, or simply the Smoluchowsky problem.

Let us stop on history of exclusively analytical solutions of the generalized Smoluchowsky' problem.

For simple (one-nuclear) rarefied gas with a constant frequency of collisions of molecules the analytical solution of the generalized of Smoluchowsky' problems it is received in [4].

In [5] the generalized of Smoluchowsky' problem was analytically solved for simple rarefied gas with frequency of collisions the molecules, linearly depending on the module of molecular velocity. In [6] the problem about strong evaporation (condensation) with constant frequency of collisions has been analytically solved.

Let us notice, that for the first time the problem about temperature jump with frequency of collisions of molecules, linearly depending on the module molecular velocity, was analytically solved by Cassel and Williams in work [7] in 1972.

Then in works [8, 9, 10] the generalized Smoluchowsky' problem also analytical solution for case of multinuclear (molecular) gases has been received.

In works [11, 12, 13] the problem about behaviour of the quantum Boze-gas at low temperatures (similar to the temperature jump problem

for electrons in metal) is considered. We used the kinetic equation with excitation fonons agrees to N.N. Bogolyubov.

In works [14, 15] the problem about temperature jump for electrons of degenerate plasmas in metal has been solved.

In work [16] the analytical solution of the Smoluchowsky' problem for quantum gases it has been received.

In work of Cercignani and Frezzotti [17] the Smoluchowsky' problem it was considered with use of the one-dimensional kinetic equations. The full analytical solution of Smoluchowsky' problem with use of Cercignani—Frezzotti equation it has been received in work [18].

At the same time there is an unresolved problem about temperature jump and concentration with use of the BGK-equation with arbitrary dependence of frequency on velocity, in spite of on obvious importance of the decision of a problem in similar statement.

In the present work attempt to promote in this direction is made. Here the case of the affine dependence of collision frequency on molecular velocity in models of one-dimensional gas is considered. Model of one-dimensional gas gave the good consent with the results devoted to the three-dimensional gas [18].

Let us start with statement problem. Then we will give the solution of the Smoluchowsky' problem for the one-dimensional kinetic equation with frequency of collisions, affine depending on the module of molecular velocity.

1. Statement of the problem and the basic equations

Let us start with statement of a problem Smoluchowsky for the one-dimensional kinetic equation with frequency of collisions, affine depending on the module velocity of molecules.

Let us begin with the general statement. Let gas occupies half-space $x > 0$. The surface temperature T_s and concentration of sated steam of a surface n_0 are set. Far from a surface gas moves with some velocity u ,

being velocity of evaporation (or condensation), also has the temperature gradient

$$g_T = \left(\frac{d \ln T}{dx} \right)_{x=+\infty}.$$

It is necessary to define jumps of temperature and concentration depending on velocity and temperature gradient.

In a problem about weak evaporation it is required to define temperature and concentration jumps depending on velocity, including a temperature gradient equal to zero, and velocity of evaporation (condensation) is enough small. The last means, that

$$u \ll v_T.$$

Here v_T is the heat velocity of molecules, having order of sound velocity order,

$$v_T = \frac{1}{\sqrt{\beta_s}}, \quad \beta_s = \frac{m}{2k_B T_s},$$

m is the mass of molecule, k_B is the Boltzmann constant.

In the problem about temperature jump it is required to define temperature and concentration jumps depending on a temperature gradient, thus evaporation (condensation) velocity it is considered equal to zero, and the temperature gradient is considered as small. It means, that

$$l g_T \ll 1, \quad l = \tau v_T, \quad \tau = \frac{1}{\nu_0},$$

where l is the mean free path of gas molecules, τ is the mean relaxation time, i.e. time between two consecutive collisions of molecules.

Let us unite both problems (about weak evaporation (condensation) and temperature jump) in one. We will assume that the gradient of temperature is small (i.e. relative difference of temperature on length of mean free path is small) and the velocity of gas in comparison with sound velocity is small. In this case the problem supposes linearization and distribution function it is possible to search in the form

$$f(x, v) = f_0(v)(1 + h(x, v)),$$

where

$$f_0(v) = n_s \left(\frac{m}{2\pi k_B T_s} \right)^{1/2} \exp \left[- \frac{mv^2}{2k_B T_s} \right]$$

is the absolute Maxwellian.

Let us pass in the equation (1.1) to dimensionless velocity

$$C = \sqrt{\beta} v = \frac{v}{v_T}$$

and dimensionless coordinate

$$x' = \nu_0 \sqrt{\frac{m}{2k_B T_s}} x = \frac{x}{l}$$

The variable x' let us designate again through x .

We take the linear kinetic equation [1]

$$\begin{aligned} & \mu \frac{\partial h}{\partial x} + (1 + \sqrt{\pi} a |\mu|) h(x, \mu) = \\ & = (1 + \sqrt{\pi} a |\mu|) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} (1 + \sqrt{\pi} a |\mu'|) q(\mu, \mu', a) h(x, \mu') d\mu'. \end{aligned} \quad (1.1)$$

Here $q(\mu, \mu', a)$ is the kernel of equation,

$$q(\mu, \mu', a) = r_0(a) + r_1(a) \mu \mu' + r_2(a) (\mu^2 - \beta(a)) (\mu'^2 - \beta(a)),$$

$$\begin{aligned} r_0(a) &= \frac{1}{a+1}, & r_1(a) &= \frac{2}{2a+1}, & r_2(a) &= \frac{4(a+1)}{4a^2+7a+2}, \\ \beta(a) &= \frac{2a+1}{2(a+1)}, \end{aligned}$$

a is the arbitrary positive paramater, $0 \leq a < +\infty$.

Let us notice, that at $a \rightarrow 0$ the equation (1.1) passes in the equation

$$\mu \frac{\partial h}{\partial x} + h(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} q(\mu, \mu') h(x, \mu) d\mu \quad (1.2)$$

with kernel

$$q(\mu, \mu') = 1 + 2\mu\mu' + 2\left(\mu^2 - \frac{1}{2}\right)\left(\mu'^2 - \frac{1}{2}\right).$$

This equation is one-dimensional BGK-equation with constant frequency of collisions.

Let us consider the second limiting case of the equation (1.1). We will return to expression of frequency of collisions also we will copy it in the form

$$\nu(\mu) = \nu_0(1 + \sqrt{\pi}a|\mu|) = \nu_0 + \nu_1|\mu|,$$

where

$$\nu_1 = \sqrt{\pi}\nu_0a.$$

Let us tend ν_0 to zero. In this limit the quantity a tends to $+\infty$, because

$$a = \frac{\nu_1}{\sqrt{\pi}\nu_0}.$$

It is easy to see, that in this limit

$$\lim_{a \rightarrow +\infty} (1 + \sqrt{\pi}a|C'|)q(\mu, \mu', a) = \sqrt{\pi}|\mu'|q_1(\mu, \mu'),$$

where

$$q_1(\mu, \mu') = 1 + \mu\mu' + (\mu^2 - 1)(\mu'^2 - 1).$$

The equation (1.1) will thus be copied in the form

$$\begin{aligned} & \frac{\mu}{|\mu|} \frac{\partial h}{\partial x_1} + h(x_1, \mu) = \\ & = \int_{-\infty}^{\infty} e^{-\mu'^2} |\mu'| [1 + \mu\mu' + (\mu^2 - 1)(\mu'^2 - 1)] d\mu'. \end{aligned} \quad (1.3)$$

In this equation

$$x_1 = \nu_1 \sqrt{\beta_s} x = \frac{x}{l_1}, \quad l_1 = v_T \tau_1, \quad \tau_1 = \frac{1}{\nu_1}.$$

This equation is the one-dimensional kinetic equation with the frequency of collisions proportional to the module of the molecular velocity.

2. Kinetic equation with constant collision frequency.

Statement of boundary problem

Rectilinear substitution it is possible to check up, that the kinetic equation (1.1) has following four private solutions

$$\begin{aligned} h_0(x, \mu) &= 1, \\ h_1(x, \mu) &= \mu, \\ h_2(x, \mu) &= \mu^2, \\ h_3(x, \mu) &= \left(\mu^2 - \frac{3}{2}\right)(x - \mu). \end{aligned}$$

Let us consider, that molecules are reflected from a wall purely diffusively, i.e. they are reflected from a wall with Maxwell distribution by velocities, i.e.

$$f(x, v) = f_0(v), \quad v_x > 0.$$

From here we receive for function $h(x, C)$ condition

$$h(0, \mu) = 0, \quad \mu > 0. \quad (2.1)$$

Condition (2.1) is the first boundary condition to the equation (1.2).

For asymptotic distribution of Chapmen—Enskog we will search in the form of a linear combination of its partial solutions with unknown coefficients

$$h_{as}(x, \mu) = A_0 + A_1\mu + A_2\left(\mu^2 - \frac{1}{2}\right) + A_3\left(\mu^2 - \frac{3}{2}\right)(x - \mu). \quad (2.2)$$

We consider the distribution of number density

$$n(x) = \int_{-\infty}^{\infty} f(x, v) dv = \int_{-\infty}^{\infty} f_0(v)(1 + h(x, v)) dv = n_0 + \delta n(x).$$

Here

$$n_0 = \int_{-\infty}^{\infty} f_0(v) dv, \quad \delta n(x) = \int_{-\infty}^{\infty} f_0(v) h(x, v) dv.$$

From here we receive that

$$\frac{\delta n(x)}{n_0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(x, \mu) d\mu.$$

We denote

$$n_e = n_0 \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} (1 + h_{as}(x=0, \mu)) d\mu.$$

From here we receive that

$$\varepsilon_n \equiv \frac{n_e - n_0}{n_0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h_{as}(x=0, \mu) d\mu. \quad (2.3)$$

The quantity ε_n is the unknown jump of concentration. Substituting (2.2) in (2.3), we find, that

$$\varepsilon_n = A_0. \quad (2.4)$$

From definition of dimensional velocity of gas

$$u(x) = \frac{1}{n(x)} \int_{-\infty}^{\infty} f(x, v) v dv$$

we receive, that in linear approximation dimensional mass velocity is equal

$$U(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(x, \mu) \mu d\mu.$$

Setting "far from a wall" velocity of evaporation (condensation), let us write

$$U = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h_{as}(x, \mu) \mu d\mu. \quad (3.5)$$

Substituting in (2.5) distribution (2.2), we receive, that

$$A_1 = 2U. \quad (2.6)$$

Let us consider temperature distribution

$$T(x) = \frac{2}{kn(x)} \int_{-\infty}^{\infty} \frac{m}{2} (v - u_0(x))^2 f(x, v) dv.$$

From here we find, that

$$\begin{aligned}\frac{\delta T(x)}{T_0} &= -\frac{\delta n}{n_0} + \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\mu^2} h(x, \mu) \mu^2 d\mu = \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\mu^2} h(x, \mu) \left(\mu^2 - \frac{1}{2}\right) d\mu.\end{aligned}$$

From here follows, that at $x \rightarrow +\infty$ asymptotic distribution is equal

$$\frac{\delta T_{as}(x)}{T_0} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\mu^2} h_{as}(x, \mu) \left(\mu^2 - \frac{1}{2}\right) d\mu. \quad (2.7)$$

Setting of the gradient of temperature far from a wall means, that distribution of temperature looks like

$$T(x) = T_e + \left(\frac{dT}{dx}\right)_{x=+\infty} \cdot x = T_e + G_T x,$$

where

$$G_T = \left(\frac{dT}{dx}\right)_{+\infty}.$$

This distribution we will present in the form

$$T(x) = T_s \left(\frac{T_e}{T_s} + g_T x\right) = T_s \left(1 + \frac{T_e - T_s}{T_s} + g_T x\right), \quad x \rightarrow +\infty,$$

where

$$g_T = \left(\frac{d \ln T}{dx}\right)_{x=+\infty},$$

or

$$T(x) = T_s (1 + \varepsilon_T + g_T x), \quad x \rightarrow +\infty,$$

where

$$\varepsilon_T = \frac{T_e - T_s}{T_s}$$

is the unknown temperature jump.

From expression (2.7) is visible, that relative change of temperature far from a wall is described by linear function

$$\frac{\delta T_{as}(x)}{T_s} = \frac{T(x) - T_s}{T_s} = \varepsilon_T + g_T x, \quad x \rightarrow +\infty \quad (2.8)$$

Substituting (2.2) in (2.7), we receive, that

$$\frac{\delta T_{as}(x)}{T_s} = A_2 + A_3 x. \quad (2.10)$$

Comparing (2.7) and (2.10), we find

$$A_2 = \varepsilon_T, \quad A_3 = g_T.$$

So, asymptotic function of Chapmen—Enskog' distribution is constructed

$$h_{as}(x, \mu) = \varepsilon_n + \varepsilon_T + 2U\mu + \left(\mu^2 - \frac{3}{2}\right)[\varepsilon_T + g_T(x - \mu)].$$

Now we will formulate the second boundary condition to the equation (1.2)

$$h(x, \mu) = h_{as}(x, \mu) + o(1), \quad x \rightarrow +\infty. \quad (2.11)$$

Now we will formulate the basic boundary problem, which is generalized Smoluchowsky' problem. This problem consists in finding of such solution of the kinetic equation (2.2) which satisfies to boundary conditions (2.1) and (2.11).

3. Eigenvalues and eigenfunctions

Seperation of variables in the equation (1.2), taken in the form Разделение переменных в уравнении (1.2), взятое в виде

$$h_\eta(x, \mu) = \exp\left(-\frac{x}{\eta}\right)\Phi(\eta, \mu), \quad \eta \in \mathbb{C}, \quad (3.1)$$

reduces this equation to the characteristic

$$(\eta - \mu)\Phi(\eta, \mu) = \frac{\eta}{\sqrt{\pi}}n_0(\eta) + \frac{2\eta}{\sqrt{\pi}}\mu n_1(\eta) + \frac{2\eta}{\sqrt{\pi}}\left(\mu^2 - \frac{1}{2}\right)n_2(\eta), \quad (3.2)$$

where $\eta, \mu \in (-\alpha, +\alpha)$,

$$n_0(\eta) = \int_{-\infty}^{\infty} e^{-\mu^2} \Phi(\eta, \mu) d\mu, \quad n_1(\eta) = \int_{-\infty}^{\infty} e^{-\mu^2} \mu \Phi(\eta, \mu) d\mu,$$

$$n_2(\eta) = \int_{-\infty}^{\infty} e^{-\mu'^2} \mu^2 \Phi(\eta, \mu) d\mu$$

are the zeroes, first and second moments of eigenfunction $\Phi(\eta, \mu)$.

Multiplying the characteristic equation (3.1) on $e^{-\mu'^2}$ and integrating on all real axis, we receive, that

$$n_1(\eta) \equiv 0.$$

Multiplying the characteristic equation (3.1) on $\mu' e^{-\mu'^2}$ and integrating on all real axis, we receive, that

$$n_2(\eta) \equiv 0.$$

We obtain the characteristic equation

$$(\eta - \mu) \Phi(\eta, \mu) = \frac{\eta}{\sqrt{\pi}} \left(\frac{3}{2} - \mu^2 \right) \int_{-\infty}^{\infty} e^{-\mu'^2} \Phi(\eta, \mu) d\mu.$$

Let us accept further the following normalization condition for the eigenfunctions $\Phi(\eta, \mu)$:

$$n_0(\eta) \equiv \int_{-\infty}^{\infty} e^{-\mu'^2} \Phi(\eta, \mu) d\mu = 1. \quad (3.3)$$

Now the characteristic equation becomes

$$(\eta - \mu) \Phi(\eta, \mu) = \frac{\eta}{\sqrt{\pi}} \left(\frac{3}{2} - \mu^2 \right). \quad (3.4)$$

Eigenfunctions of the continuous spectrum filling by the continuous fashion the interval $(-\infty, \infty)$, We find [19] in space of the generalized functions

$$\Phi(\eta, \mu) = \frac{\eta}{\sqrt{\pi}} \left(\frac{3}{2} - \mu^2 \right) P \frac{1}{\eta - \mu} + e^{\eta^2} \lambda(\eta) \delta(\eta - \mu), \quad \eta \in (-\infty, \infty). \quad (3.5)$$

Here $\lambda(\eta)$ is the dispersion fuction, defined by equation (3.3), Px^{-1} is the distribution, meaning principal value of integral at intrgration x^{-1} , $\delta(x)$ is the Dirac function,

$$\lambda(z) = 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2} \frac{3/2 - \tau^2}{\tau - z} d\tau = -\frac{1}{2} + \left(\frac{3}{2} - z^2\right) \lambda_0(z),$$

$$\lambda_0(z) = 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2} \frac{d\tau}{\tau - z} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\tau^2} \tau \frac{d\tau}{\tau - z}.$$

Apparently from the solution of the characteristic equation, continuous spectrum of the characteristic equation is the set

$$\sigma_{\mu} = \{\eta : -\infty < \eta < +\infty\}.$$

By definition by the discrete spectrum of the characteristic equation is set of zero of dispersion function.

Expanding dispersion function in Laurent series in a vicinity infinitely remote point, we are convinced, that it in this point has zero of the fourth order. Applying an argument principle from the theory of functions complex variable, it is possible to show, that other zero, except $z_i = \infty$, dispersion function not has. Thus, the discrete spectrum of the characteristic equations consists of one point $z_i = \infty$, which multiplicity is equal four,

$$\sigma_d = \{z_i = \infty\}.$$

To point $z_i = \infty$, as to a 4-fold point of the discrete spectrum, corresponds the following four discrete (partial) solutions the kinetic decision (1.2): $h_0(x, \mu)$, $h_1(x, \mu)$, $h_3(x, \mu)$ and $h_3(x, \mu)$.

Let us result Sokhotsky formulas for the difference and the sum of the boundary values of dispersion function from above and from below on the cut $(-\infty, +\infty)$:

$$\lambda^+(\mu) - \lambda^-(\mu) = 2\sqrt{\pi}i\mu e^{-\mu^2} \left(\frac{3}{2} - \mu^2\right), \quad \mu \in (-\infty, +\infty),$$

and

$$\frac{\lambda^+(\mu) + \lambda^-(\mu)}{2} = -\frac{1}{2} + \left(\frac{3}{2} - \mu^2\right) \lambda_0(\mu), \quad \mu \in (-\infty, +\infty).$$

On the real axis function $\lambda_0(\mu)$ is calculated on to the formula

$$\lambda_0(\mu) = 1 - 2\mu^2 \int_0^1 e^{-\mu^2(1-t^2)} dt.$$

4. Homogeneous boundary Riemann problem

Here we will consider homogeneous boundary Riemann problem from theories of functions complex variable which is required further. This problem consists in the finding of such function $X(z)$, which is analytical in a complex plane, cut along the real positive half-axis $\mathbb{C}' = \mathbb{C} \setminus \mathbb{R}^+$.

Boundary values of this function from above and from below on the real half-axis satisfy to the boundary condition

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \quad \mu > 0. \quad (4.1)$$

We note that

$$|\lambda^+(\mu)| = |\lambda^-(\mu)|, \quad \lambda^+(\mu) = \overline{\lambda^-(\mu)}, \quad \mu \in (-\infty, +\infty).$$

Let us enter the principal value of argument $\theta(\mu) = \arg \lambda^+(\mu)$, defined in the cut plane \mathbb{C}' and fixed in zero by the condition $\theta(0) = 0$. Then

$$\lambda^+(\mu) = |\lambda^+(\mu)| e^{i\theta(\mu)}, \quad \lambda^-(\mu) = |\lambda^-(\mu)| e^{-i\theta(\mu)}.$$

Noe the problem (4.1) will be rewritten in the form

$$\frac{X^+(\mu)}{X^-(\mu)} = e^{2i\theta(\mu)}, \quad \mu > 0. \quad (4.2)$$

Taking the logarithm of the problem (4.2), we receive following nume-
rable family of problems of the finding of analytical function on its zero
jump on the positive real half-axis $\mathbb{R}^+ = \{\mu : \mu > 0\}$:

$$\ln X^+(\mu) - \ln X^-(\mu) = 2i\theta(\mu) + 2\pi ik, \quad k \in \mathbb{Z}, \quad \mu > 0. \quad (4.3)$$

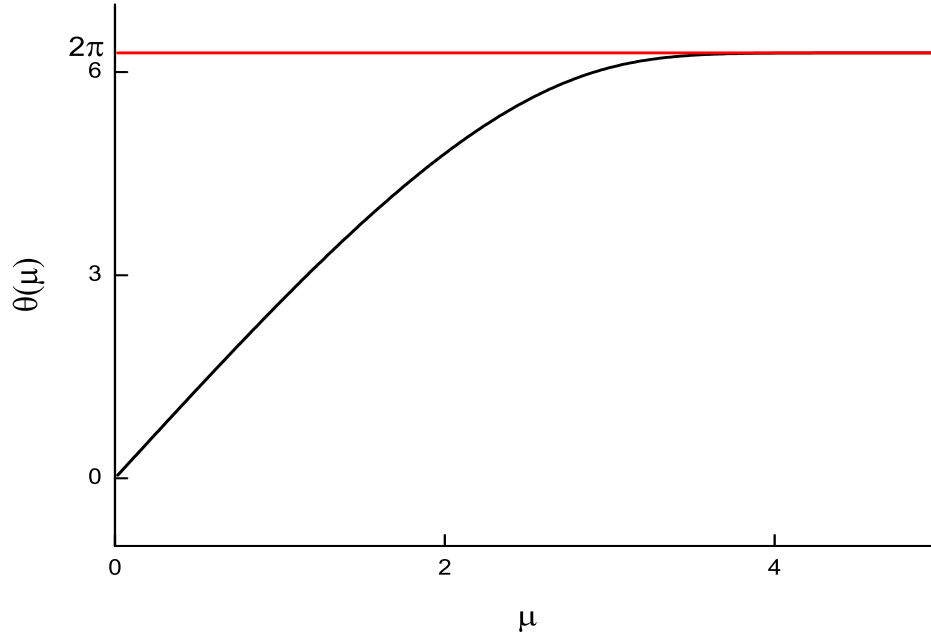


Рис. 1. The angle $\theta = \theta(\mu)$ monotonously increases from 0 to 2π .

The solution of problems (4.3) is expressed by integral of Cauchy type

$$\ln X(z) = \frac{1}{\pi} \int_0^{\infty} \frac{\theta(\mu) + k\pi}{\mu - z} d\mu, \quad k \in \mathbb{Z}.$$

Let us notice, that the angle $\theta(\mu)$ is on a semiaxis \mathbb{R}^+ monotonously increasing function from 0 to 2π . It means, that index of coefficient $G(\mu) = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}$ of homogeneous Riemann problem (5.1) on the positive real half-axis is equal to unit

$$\varkappa = \varkappa(G) = \frac{1}{2\pi} \left[\arg G(\mu) \right] \Big|_0^{\infty} = 1.$$

From here follows, that among family of solutions (4.3) only one (at $k = -2$) is expressed by the converging integral of Cauchy type

$$\ln X(z) = \frac{1}{\pi} \int_0^{\infty} \frac{\theta(\mu) - 2\pi}{\mu - z} d\mu. \quad (4.4)$$

We denote further

$$V(z) = \ln X(z),$$

whence

$$X(z) = e^{V(z)}.$$

Let us redefine the received solution as follows

$$X(z) = \frac{1}{z^2} e^{V(z)}. \quad (4.5)$$

Let us notice, that the solution (4.5) is bounded function in a vicinity the point $z = 0$. Really, at $z \rightarrow 0$ it is had

$$V(z) = -\frac{\theta(0) - 2\pi}{\pi} \ln z + O(z), \quad z \rightarrow 0,$$

where $O(z)$ is the bounded function in a vicinity the point $z = 0$. Hence, in a vicinity of the point $z = 0$ function $X(z) = e^{O(z)}$ is the bounded function.

5. Analytical splution of the boundary problem for kinetic equation with constant collision frequency

Here we will prove the theorem about the analytical solution of the basic boundary problem (1.2), (2.1) and (2.11).

THEOREM. *Boundary problem (1.2), (2.1) and (2.11) has the unique decision, representable in the form of the sum linear combinations of discrete (partial) solutions of this equation and integral on the continuous spectrum from eigenfunctions correponding to the continuous spectrum*

$$h(x, \mu) = h_{as}(x, \mu) + \int_0^\infty \exp\left(-\frac{x}{\eta}\right) \Phi(\eta, \mu) A(\eta) d\eta. \quad (5.1)$$

In equality (5.1) ε_n and ε_T are unknown coefficient (discrete spectrum), U and g_T are the given qualities, $A(\eta)$ is the unknown function (coefficient of the continuous spectrum).

Coefficients of discrete and continuous spectra are subject to finding from boundary conditions.

Expansion (5.1) it is possible to present in the explicit form in classical sense

$$h(x, \mu) = \varepsilon_n + \varepsilon_T + 2U\mu + \left(\mu^2 - \frac{3}{2}\right)[\varepsilon_T + g_T(x - \mu)] + e^{\mu^2 - x/\mu} \lambda(\mu) A(\mu) + \left(\frac{3}{2} - \mu^2\right) \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-x/\eta} \frac{\eta A(\eta)}{\eta - \mu} d\eta. \quad (5.1')$$

PROOF. Let us substitute expansion (5.1) in the boundary condition (2.1). We receive the integral equation

$$h_{as}(0, \mu) + \int_0^\infty \Phi(\eta, \mu) A(\eta) d\eta = 0, \quad 0 < \mu < \infty.$$

In the explicit form this equation looks like

$$h_{as}(0, \mu) + e^{\mu^2} \lambda(\mu) A(\mu) + \left(\frac{3}{2} - \mu^2\right) \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\eta A(\eta)}{\eta - \mu} d\eta = 0, \quad 0 < \mu < \infty. \quad (5.2)$$

Here

$$h_{as}(0, \mu) = \varepsilon_n + \varepsilon_T + 2U\mu + \left(\mu^2 - \frac{3}{2}\right)(\varepsilon_T - g_T\mu).$$

Let us enter auxiliary function

$$N(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\eta A(\eta)}{\eta - z} d\eta, \quad (5.3)$$

for which according to formulas Sokhotsky it is had

$$N^+(\mu) - N^-(\mu) = 2\sqrt{\pi}i\mu A(\mu), \quad 0 < \mu < \infty, \quad (5.4)$$

$$\frac{N^+(\mu) + N^-(\mu)}{2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\eta A(\eta)}{\eta - \mu} d\eta, \quad 0 < \mu < \infty. \quad (5.5)$$

Let us transform the equation (5.2), considering formulas Sokhotsky for dispersion function and according to equalities (5.4) and (5.5). We receive non-uniform the boundary Riemann condition

Considering formulas Sokhotsky for dispersive function, Let's transform the equation (5.6) to a non-uniform regional problem Римана:

$$\begin{aligned} & \lambda^+(\mu) \left[\left(\frac{3}{2} - \mu^2 \right) N^+(\mu) + h_{as}(0, \mu) \right] - \\ & - \lambda^-(\mu) \left[\left(\frac{3}{2} - \mu^2 \right) N^-(\mu) + h_{as}(0, \mu) \right] = 0, \quad 0 < \mu < \infty. \end{aligned} \quad (5.6)$$

Let us consider the corresponding homogeneous boundary Riemann problem

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \quad 0 < \mu < \infty. \quad (5.7)$$

The solution of this problem which is bounded and not disappearing in points $z = 0$ and $z = \alpha$ it is resulted in the previous item

$$X(z) = \frac{1}{z^2} \exp V(z), \quad (5.8)$$

where

$$V(z) = \frac{1}{\pi} \int_0^\infty \frac{\theta(\mu) - 2\pi}{\mu - z} d\mu, \quad (6.9)$$

$\theta(\mu) = \arg \lambda^+(\mu)$ is the principal value of argument, fixed by condition $\theta(0) = 0$.

Let us transform the problem (5.6) by means of the homogeneous problem (5.7) to the problem of finding of analytical function on its jump on the cut

$$\begin{aligned} & X^+(\mu) \left[\left(\frac{3}{2} - \mu^2 \right) N^+(\mu) + h_{as}(0, \mu) \right] = \\ & = X^-(\mu) \left[\left(\frac{3}{2} - \mu^2 \right) N^-(\mu) + h_{as}(0, \mu) \right], \quad 0 < \mu < \infty. \end{aligned} \quad (5.10)$$

Let us find singularities of the boundary condition (5.10). Considering behaviour of the functions entering into boundary condition (5.10), we

receive the common solution corresponding to boundary problem

$$\left(z^2 - \frac{3}{2}\right)N(z) = h_{as}(0, z) + \frac{C_0 + C_1 z}{X(z)}, \quad (5.11)$$

where C_0 and C_1 are arbitrary constants, and

$$h_{as}(0, z) = \varepsilon_n + \varepsilon_T + 2U\mu + \left(z^2 - \frac{3}{2}\right)(\varepsilon_T - g_T z).$$

Let us notice, that the solution (5.11) has in infinitely removed point $z = \infty$ a pole of the third order, while function $N(z)$, defined by equality (5.3), has in this point a pole the first order.

That the solution (5.11) could be accepted in quality of function $N(z)$, defined by equality (5.3), we will lower order of a pole at the solution (5.11) from three to unit.

Then let us equate values of the left and right parts of equality (5.11) in points of the real axis $\mu_{1,2} = \pm\sqrt{3/2}$.

Decomposition is required to us

$$V(z) = \frac{V_1}{z} + \frac{V_2}{z^2} + \dots, \quad z \rightarrow \infty.$$

Here

$$V_n = -\frac{1}{\pi} \int_0^\infty \tau^{n-1} [\theta(\tau) - 2\pi] d\tau, \quad n = 1, 2, \dots$$

Lowering an order of pole on two units in infinitely removed point at the solution (5.11), we find, that

$$C_0 = V_1 g_T - \varepsilon_T,$$

$$C_1 = g_T.$$

The pole of function in the point $\mu_1 = \sqrt{3/2}$ is eliminated by two limiting conditions from above and from below the real axis, for this point lays on a cut (the real axis)

$$C_0 + C_1 \mu_1 + X^+(\mu_1)(\varepsilon_n + \varepsilon_T + 2U\mu_1) = 0, \quad (5.12)$$

and

$$C_0 + C_1\mu_1 + X^-(\mu_1)(\varepsilon_n + \varepsilon_T + 2U\mu_1) = 0, \quad (5.13)$$

The point $\mu_2 = -\mu_1$ does not belong to the cut, therefore we receive

$$C_0 - C_1\mu_1 + X(-\mu_1)(\varepsilon_n + \varepsilon_T - 2U\mu_1) = 0. \quad (5.14)$$

We take a half-sum of conditions (5.12) and (5.13)

$$C_0 + C_1\mu_1 + \frac{X^+(\mu_1) + X^-(\mu)}{2}(\varepsilon_n + \varepsilon_T + 2U\mu_1) = 0 \quad (5.15)$$

We note that

$$X^\pm(\mu_1) = \frac{1}{\mu_1^2} e^{V^\pm(\mu_1)},$$

where

$$V^\pm(\mu_1) = V(\mu_1) \pm i[\theta(\mu_1) - 2\pi].$$

Hence

$$\frac{X^+(\mu_1) + X^-(\mu)}{2} = X(\mu_1).$$

Taking into account this equality from the equations (5.14) and (5.15) it is received expressions of required qualities of jump of temperature and jump concentration

$$\varepsilon_T = g_T \left[V_1 - \mu_1 \frac{X(\mu_1) + X(-\mu_1)}{X(\mu_1) - X(-\mu_1)} \right] - 4U\mu_1 \frac{X(\mu_1)X(-\mu_1)}{X(\mu_1) - X(-\mu_1)} \quad (5.16)$$

and

$$\begin{aligned} \varepsilon_n = g_T & \left[V_1 + \mu_1 \frac{X(\mu_1) + X(-\mu_1) - 2}{X(\mu_1) - X(-\mu_1)} \right] - \\ & - 2U\mu_1 \frac{X(\mu_1) + X(-\mu_1) - 2X(\mu_1)X(-\mu_1)}{X(\mu_1) - X(-\mu_1)}. \end{aligned} \quad (5.17)$$

Coefficient of continuous spectrum $A(\eta)$ can be found on the basis of the formula Sokhotsky (5.4) and formulas of the difference of boundary values $N(z)$, received with the help solutions (5.11)

$$N^+(\mu) - N^-(\mu) = \frac{C_0 + C_1\mu}{\mu^2 - 3/2} \left[\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)} \right]. \quad (5.18)$$

From equalities (5.4) and (5.18) we find coefficient of the continuous spectrum

$$2\sqrt{\pi}i\eta A(\eta) = \frac{g_T(V_1 + \eta) - \varepsilon_T}{\eta^2 - 3/2} \left[\frac{1}{X^+(\eta)} - \frac{1}{X^-(\eta)} \right]. \quad (5.19)$$

We note that

$$\frac{1}{X^+(\eta)} - \frac{1}{X^-(\eta)} = -\frac{2i}{X(\eta)} \sin \theta(\eta).$$

By means of this equality coefficient of the continuous spectrum (5.19) it is definitively equal

$$\eta A(\eta) = -\frac{(V_1 + \eta)g_T - \varepsilon_T}{\sqrt{\pi}X(\eta)(\eta^2 - 3/2)} \sin \theta(\eta).$$

So, all coefficients of expansion (5.1) are established. On to construction, expansion (5.1) satisfies to boundary conditions (2.1) and (2.11). That fact, that expansion (5.1) satisfies to the equation (1.2), it is checked directly.

Uniqueness of decomposition (5.1) is proved by a method from the opposite. The theorem is proved.

6. Temperature jump and weak evaporation (condensation). Numerical calculations

Numerical calculations of coefficients V_n lead to the following results

$$V_1 = 2.6470 \dots, \quad V_2 = 2.5, \quad V_3 = 3.7153 \dots,$$

and also

$$\mu_1 = 1.2247 \dots, \quad X(\mu_1) = 3.8483 \dots, \quad X(-\mu_1) = 0.1732 \dots.$$

Now it is required to us following

THEOREM. For dispersion function $\lambda(z)$ takes place the following factorization formula

$$\begin{aligned}\lambda(z) &= -\frac{3}{4}X(z)X(-z), & z \in \mathbb{C}', \\ \lambda^+(\mu) &= -\frac{3}{4}X^+(\mu)X(-\mu), & \mu > 0, \\ \lambda^-(\mu) &= -\frac{3}{4}X(\mu)X^+(-\mu), & \mu < 0.\end{aligned}$$

PROOF. This theorem is proved in the same way, as well as the proof of similar theorems in our works [3].

By means of this theorem it is found exact value

$$X(\mu_1)X(-\mu_1) = \frac{2}{3}.$$

We rewrite these formulas (5.16) and (5.17) in the form

$$\varepsilon_T = K_{TT}g_T + K_{TU}(2U), \quad \varepsilon_n = K_{nT}g_T + K_{nU}(2U).$$

Here

$$\begin{aligned}K_{TT} &= V_1 - \mu_1 \frac{X(\mu_1) + X(-\mu_1)}{X(\mu_1) - X(-\mu_1)}, \\ K_{TU} &= -2\mu_1 \frac{X(\mu_1)X(-\mu_1)}{X(\mu_1) - X(-\mu_1)}, \\ K_{nT} &= V_1 + \mu_1 \frac{X(\mu_1) + X(-\mu_1) - 2}{X(\mu_1) - X(-\mu_1)}, \\ K_{nU} &= -\mu_1 \frac{X(\mu_1) + X(-\mu_1) - 2X(\mu_1)X(-\mu_1)}{X(\mu_1) - X(-\mu_1)}.\end{aligned}$$

Now it is easy to find that

$$\begin{aligned}K_{TT} &= 1.3068, & K_{TU} &= -0.4443, \\ K_{nT} &= 3.3207, & K_{nU} &= -0.8958.\end{aligned}$$

Hence, coefficient of jump of temperature and jump of concentration are calculated under formulas

$$\varepsilon_T = 1.3068g_T - 0.4443(2U), \tag{6.1}$$

and

$$\varepsilon_n = -3.3207g_T - 0.8958(2U). \quad (6.2)$$

7. Limiting transition in the general formulas

Here we will show, that if we will make limiting transition in the general formulas (7.8) and (7.9) from [2] at $a \rightarrow 0$, we in accuracy let us receive formulas (5.16) and (5.17). We will remind, that formulas (7.8) and (7.9) for temperature and concentration jump are received for the case frequencies of collisions, affine depending on the module of velocity.

We will do this transition for temperature jump in the case of weak evaporations (i.e. for the case $g_T = 0$).

Let us transform the formula (5.16) to the form

$$\varepsilon_T = -2U \frac{1}{\frac{1}{2\mu_1 X(-\mu_1)} - \frac{1}{2\mu_1 X(\mu_1)}}. \quad (7.1)$$

From formula (7.8) from [2] we receive

$$\varepsilon_T = -2U \frac{1}{V_1 + \lim_{a \rightarrow 0} K_1}, \quad (7.2)$$

where

$$K_1 = \frac{1}{2\pi i} \int_0^{1/a} \left(\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)} \right) \frac{d\mu}{Q(\mu, \mu)}.$$

For coincidence of equalities (7.1) and (7.2) it is required to prove equality

$$\lim_{a \rightarrow 0} K_1 = -V_1 + \frac{1}{2\mu_1 X(-\mu_1)} - \frac{1}{2\mu_1 X(\mu_1)} \quad (7.3)$$

We note that in considering case at $a \rightarrow 0$: $r_2(a) \rightarrow 2$, $\beta(a) \rightarrow \frac{1}{2}$, $\omega(a) \rightarrow 0$, $r_0(a) \rightarrow 1$, $r_1(a) \rightarrow 2$. Hence,

$$\Lambda_2(\mu) \equiv 0, \quad \Lambda_0(\mu) \equiv 1, \quad Q(\mu, \mu) = \frac{3}{2} - \mu^2.$$

Therefore

$$\lim_{a \rightarrow 0} = \frac{1}{2\pi i} \int_0^\infty \left(\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)} \right) \frac{d\mu}{Q(\mu, \mu)}.$$

We form the difficult contour Γ_R , consisting of the external circles $|z| = R$ with radius $R = 1/a$, and two internal circles $|z \pm \mu_1| = a$ with radiuses $r = a$.

Under Cauchy theorem

$$\oint_{\Gamma_R} \frac{d\mu}{X(\mu)Q(\mu, \mu)} = 0.$$

We denote

$$\lim_{a \rightarrow 0} K_1 = I.$$

Then from the previous equality it is received

$$I = I_1 + I_2 - I_3,$$

where

$$\begin{aligned} I_1 &= \lim_{a \rightarrow 0} \frac{1}{2\pi i} \oint_{|\mu + \mu_1| = a} \frac{d\mu}{X(\mu)Q(\mu, \mu)}, \\ I_2 &= \lim_{a \rightarrow 0} \frac{1}{2\pi i} \oint_{|\mu - \mu_1| = a} \frac{d\mu}{X(\mu)Q(\mu, \mu)}, \\ I_3 &= \lim_{a \rightarrow 0} \frac{1}{2\pi i} \oint_{|\mu| = 1/a} \frac{d\mu}{X(\mu)Q(\mu, \mu)}. \end{aligned}$$

It is easy to see, that

$$I_1 = \frac{1}{Q'(\mu, \mu)X(\mu)} \Big|_{\mu = -\mu_1} = \frac{1}{2\mu_1 X(\mu_1)},$$

in the same way

$$I_2 = -\frac{1}{2\mu_1 X(\mu_1)}.$$

For calculation of integral I_3 we will spread out its subintegral function by Laurent series in a vicinity of infinitely remote point and let us present it in the form

$$\frac{1}{Q(z, z)X(z)} = -1 + \frac{V_1}{z} + \varphi(z), \quad \mu \rightarrow \infty,$$

where

$$\varphi(z) = O(z^{-2}), \quad z \rightarrow \infty.$$

Hence, this integral equals

$$\frac{1}{2\pi i} \oint_{|\mu|=1/a} \frac{d\mu}{X(\mu)Q(\mu, \mu)} = V_1 + \frac{1}{2\pi i} \oint_{|\mu|=1/a} \varphi(\mu) d\mu.$$

The limit of last integral is equal in this equality to zero at $a \rightarrow 0$ owing to previous asymptotic, therefore $I_3 = V_1$.

So, equality (7.3) is established.

8. Distribution of macroparameters of gas in "half-space"

Let us consider distribution of concentration, mass velocity and temperature depending on coordinate x .

Let us begin with concentration distribution (numerical density)

$$\begin{aligned} \frac{\delta n(x)}{n_0} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(x, \mu) d\mu = \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} \left[h_{as}(x, \mu) + \int_0^{\infty} e^{-x/\eta} \Phi(\eta, \mu) A(\eta) d\eta \right] d\mu = \\ &= \varepsilon_T - g_T x + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-x/\eta} d\eta \int_{-\infty}^{\infty} e^{-\mu^2} \Phi(\eta, \mu) A(\eta) d\mu. \end{aligned}$$

Having taken advantage of the normalizing equality (3.3), we receive

$$\frac{\delta n(x)}{n_0} = \varepsilon_T - g_T x + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-x/\eta} A(\eta) d\eta.$$

Let us transform coefficient of the continuous spectrum. Noticing, that

$$\sin \theta(\eta) = \frac{\sqrt{\pi}\eta e^{-\eta^2}(3/2 - \eta^2)}{|\lambda^+(\eta)|}.$$

Hence,

$$A(\eta) = \frac{(V_1 + \eta)g_T - \varepsilon_T}{X(\eta)|\lambda^+(\eta)|}e^{-\eta^2}.$$

Thus, we come to following distribution of concentration

$$\begin{aligned} \frac{\delta n(x)}{n_0} = & [K_{TT}(1 - m_0(x)) - x + V_1 m_0(x) + m_1(x)]g_T + \\ & + K_{TU}(1 - m_0(x))(2U). \end{aligned}$$

Here

$$m_k(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-x/\eta - \eta^2} \eta^k d\eta}{X(\eta)|\lambda^+(\eta)|}, \quad k = 0, 1.$$

Mass velocity $U(x)$ is equal everywhere at $x > 0$ to given on infinity quantity of velocity, i.e. $U(x) \equiv U$. Really, we have

$$\begin{aligned} U(x) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-\mu^2} h(x, \mu) \mu d\mu = \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-\mu^2} \left[h_{as}(x, \mu) + \int_0^\infty e^{-x/\eta} \Phi(\eta, \mu) A(\eta) d\eta \right] \mu d\mu. \end{aligned}$$

From here we obtain that

$$U(x) = U + \int_0^\infty e^{-x/\eta} A(\eta) d\eta \int_{-\infty}^\infty e^{-\mu^2} \mu \Phi(\eta, \mu) d\mu \equiv U,$$

because the first moment of eigenfunction $\Phi(\eta, \mu)$ is equal to zero as it has been shown above.

We consider the distribution of temperature

$$\frac{\delta T(x)}{T_0} = \varepsilon_T + g_T x + \frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-\mu^2} \left(\mu^2 - \frac{1}{2} \right) h(x, \mu) d\mu =$$

$$= \varepsilon_T + g_T x + \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x/\eta} A(\eta) d\eta \int_{-\infty}^{\infty} e^{-\mu^2} \left(\mu^2 - \frac{1}{2} \right) \Phi(\eta, \mu) d\mu.$$

Considering, that the second moment of eigenfunction $\Phi(\eta, \mu)$ is equal to zero, from here we receive the temperature distribution

$$\begin{aligned} \frac{\delta T(x)}{T_0} &= \varepsilon_T + g_T x - \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-x/\eta} A(\eta) d\eta = \\ &= [x - m_1(x) + K_{TT}(1 + m_0(x))]g_T + K_{TU}(1 + m_0(x))(2U). \end{aligned}$$

9. Conclusion

In the present work the analytical solution of boundary problems for the one-dimensional kinetic equation with constant frequency of collisions of molecules is considered. We consider the solution of the generalized Smoluchowsky problem (problems about temperature jump and weak evaporation (condensation)). Numerical calculations are done. Distribution of concentration, mass speed and temperature is received.

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